Optimal Interplanetary Trajectories for Chemically Propelled Spacecraft

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Primer vector theory is applied to trajectory optimization for high-thrust, interplanetary missions. Using a sphere-of-influence analysis, it is shown that the interplanetary problem can be solved in the same format with which the central-body problem has been treated, merely by modifying the initial and final primer vectors. For the minimum-impulse case, this reduces to applying a scalar factor to the primer, but to obtain maximum final spacecraft mass, the primer must also undergo a rotation. The rotation accounts for the fact that launch vehicle performance is a function of both hyperbolic excess velocity and launch asymptote declination. This theory is used to generate optimal interplanetary trajectories for the 1973 Mars opportunity and a comparison between the optimal results and single-plane, ballistic transfers is presented.

Introduction

PRIMER vector theory, first developed by Lawden[1] and extended to include nonoptimal as well as optimal trajectories by Lion and Handelsman,[2] has seen considerable use in the computation of minimum-Δv, impulsive space trajectories in a central gravitational field. Computer programs, e.g., those by Jezewski and Rosendaal,[3] Minkoff and Lion,[4] and Hazelrigg and Sachs,[5] demonstrate the high degree of success obtainable by this method. However, while much of this success can be attributed to the eloquent primer theory, one must not neglect the fact that this basic problem is mathematically clean. In general, one can say that a solution exists that is also often unique and nonsingular. The result is that with the use of a good iterative technique, e.g., the Fletcher-Powell algorithm,[6] solutions are found easily and economically.

One very interesting application of the above technique is determination of optimal trajectories for interplanetary missions. To do this, however, requires extension of the basic theory in two ways: 1) to include the effects of the planet's gravitational fields, and 2) since it is often of interest to maximize payload rather than minimize Δv, to account for staging and specific impulse differences between stages.

This paper extends the work of Lion and Handelsman to cover the optimal transfer of a chemically propelled (or "high-thrust") spacecraft from Earth to an orbit around another planet. The work is accomplished in two steps. First, a method is derived for computing fixed-time, minimum-Δv transfers between Earth-orbit and planet-orbit. Second, this theory is extended to account for vehicle staging and specific impulse differences via the introduction of the (fictitious) equivalent spacecraft characteristic velocity. Minimizing this quantity yields maximized spacecraft payload.

This work was prompted by the unfavorable Earth-launch geometry for the 1973, 1975, and 1977 Mars opportunities, which require excessive geocentric orbital declinations or considerably lengthened trip times. The solutions found offer very substantial improvements in trajectory geometry.

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Fundamentals of Impulsive Trajectory Optimization

The purpose of this section is to review the derivation of necessary conditions for the optimal impulsive transfer of a spacecraft in a central gravitational field. For more detail, refer to Refs. 1–5.

The calculus of variations is used to minimize the functional J where

$$J = \sum_{k=1}^{n} |\Delta v_k| \tag{1}$$

n is the number of impulses required on the optimal trajectory (to be determined) and Δv_k is the velocity increment of the kth impulse. J is subject to differential constraints—the equations of motion

$$\dot{r} = v, \qquad \dot{v} = g(r) \tag{2}$$

where r denotes position, v denotes velocity, and g is the gravitational acceleration—to endpoint constraints; e.g., for rendezvous:

$$r(0) = r_0, \quad r(t_f) = r_f \tag{3a}$$

and

$$v(0) = v_0, \quad v(t_f) = v_f \tag{3b}$$

and to state variable constraints to define the impulses

$$r(t_k^+) = r(t_k^-), \quad v(t_k^+) = v(t_k^-) + \Delta v_k \tag{4}$$

t_k^- and t_k^+ refer to times immediately before and after the kth impulse. These constraints are adjoined to J with the Lagrangian multipliers λ on velocity and, since λ obeys the second order differential equation, $\ddot{\lambda} = (\partial g/\partial r)\lambda$, $-\dot{\lambda}$ on r. While other multipliers must be used initially for the constraints (3) and (4), they are readily expressed in terms of λ and $\dot{\lambda}$.

Applying the necessary conditions for optimality as given by Mason et al.,[7] or Denham,[8] and augmented by Lion[2] yields, at an impulse,

$$\lambda_k = \hat{\eta}_k \tag{5}$$

where $\hat{\eta}_k$ is a unit vector in the direction of the kth impulse and, at all other times

$$|\lambda| < 1 \tag{6}$$

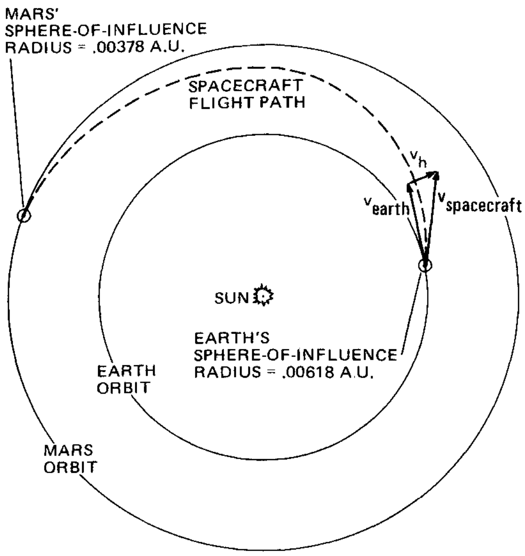

Fig. 1 Size of Earth and Mars spheres-of-influence.

Also, as an extension of Eqs. (5) and (6), since both λ and $\dot{\lambda}$ are continuous functions of time,

$$\lambda_k \cdot \dot{\lambda}_k = 0 \tag{7}$$

For nonoptimal trajectories, the adjoint variables provide the gradient of J with respect to the time and location of impulses.

$$\partial J/\partial t_k = \dot{\lambda}_k^{-} \cdot v_k^{-} - \dot{\lambda}_k^{+} \cdot v_k^{+}, \quad \partial J/\partial r_k = \dot{\lambda}_k^{+} - \dot{\lambda}_k^{-} \tag{8}$$

Note that, since r_0 and r_f are fixed, the latter of Eqs. (8) does not yield information for impulses occurring at the initial or final time.

The key to all the aforementioned is the definition of λ_k, Eq. (5). For impulses occurring at the initial and final times, this derives from (the transversality conditions)

$$\lambda_0 = \partial|\Delta v_1|/\partial v_0^{+}, \quad \lambda_f = \partial|\Delta v_n|/\partial v_f^{-} \tag{9}$$

Equations (9) will form the basis of the work to follow.

Planet-to-Planet Transfers, Minimum-Δv

The transfer of a spacecraft from an orbit around one planet to an orbit around another planet involves a multibody gravitational field in which the gravitational fields of the sun and of both planets each contribute the primary component of gravitational acceleration for some portion of the flight. Consequently, none can be neglected; however, the resulting spacecraft motion can be analytically determined within reasonable tolerances by the method of the spheres-of-influence.[9,10] Herewith, it is assumed that the motion is always inverse-square, subject to the attraction of the central body of the sphere-of-influence in which the spacecraft is moving. Further, when dealing with interplanetary flights, it is common to assume that the spheres-of-influence of the planets are of negligible size compared to interplanetary distances. See Fig. 1, where v_h is the hyperbolic excess volocity. Thus, an interplanetary flight can be approximated as a heliocentric flight from the position of the departure planet to the position of the arrival planet. But, due to the gravitational fields of the planets, the transversality conditions (9) do not yield (5). Instead, if the orientations of the planetary orbits are free to be optimized, Eq. (9) yields

$$\lambda_0 = \alpha_0 \hat{\eta}_1, \quad \lambda_f = \alpha_f \hat{\eta}_n \tag{10}$$

where α is a scale factor, always less than unity

$$\alpha = \partial|\Delta v|/\partial v_h \tag{11}$$

Now consider a planetary escape or capture maneuver. For a planar escape to a specified hyperbolic excess velocity, Hazelrigg[11] has shown that the optimal maneuver usually consists of a single tangential impulse at periapse, Fig. 2. If the escape maneuver must be nonplanar and no time constraints are imposed on flight in a sphere-of-influence, the trajectory shown in Fig. 3 can be used to approximate the planar case to any desired degree. Considering the planar case, the $|\Delta v|$ required can be written

$$|\Delta v| = [v_h^2 + 2v_c^2]^{1/2} - [2v_c^2 - v_a^2]^{1/2} \tag{12}$$

where v_c is the circular satellite velocity at periapse,

$$v_c = [\mu/r_p]^{1/2} \tag{13}$$

and v_a is the average satellite velocity

$$v_a = [2\mu/(r_p + r_a)]^{1/2} \tag{14}$$

μ is the planet's gravitational constant, and r_p and r_a are the periapse and apoapse radii, respectively. Consequently, α becomes

$$\alpha = [1 + 2v_c^2/v_h^2]^{-1/2} \tag{15}$$

This result was also obtained by Lawden.[12]

Lion[2] shows that, for the central-body field, there are two basic modes of improving a nonoptimal trajectory. Suppose the trajectory in question uses m impulses and that the interior impulse times and positions are optimized. This ensures that $\lambda_k \cdot \dot{\lambda}_k = 0$ and $|\lambda_k| = 1$ at all interior impulses [see Eqs. (5) and (7)]. Assume, however, that this trajectory violates some necessary conditions; e.g., $|\lambda| > 1$ at some time. Then the trajectory can be improved; namely, J can be reduced, by the introduction of another impulse, appropriately placed at a time when $|\lambda| > 1$, thus making an $m + 1$-impulse trajectory. A second mode, namely allowing terminal coasts (rather than impulsing at the start and/or finish

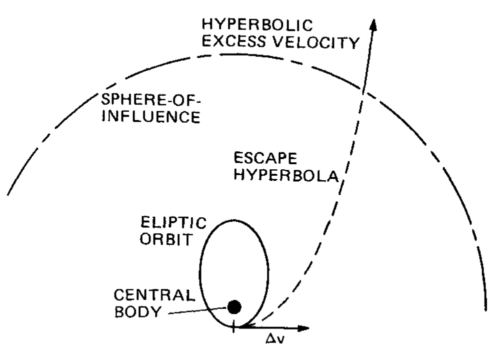

Fig. 2 Minimum-Δv planar escape maneuver.

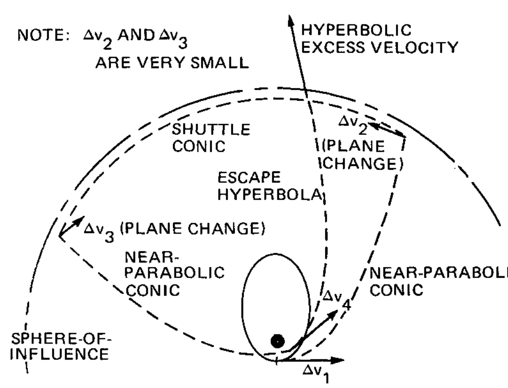

Fig. 3 Minimum-Δv escape with plane change.

of the transfer) can also improve the trajectory under certain conditions; specifically, if $|\lambda| > 1$ immediately after the initial time for an initial coast or immediately prior to the final time for a final coast. These conditions, necessary for terminal coasts, are

$$\lambda_0 \cdot \dot{\lambda}_0 > 0, \quad \lambda_f \cdot \dot{\lambda}_f < 0 \tag{16}$$

where it is assumed that $|\lambda_0| = |\lambda_f| = 1$. Clearly, when the vehicle is escaping from a planet, $|\lambda_0| < 1$ and when the vehicle arrives at a planet, $|\lambda_f| < 1$. It follows, noting that $|\lambda|$ is a continuous function of time, that $|\lambda|$ immediately after the start or before the finish of a planet-to-planet transfer will never exceed unity; hence, initial or final coasts are never called for when the problem is solved in the above format. Strictly speaking, this is incorrect and what has happened is that the trajectory has been constrained to depart at the initial time and arrive at the final time. To relax this constraint, if desired, one must allow for coasts in the planetary orbits. Consequently, conditions (16) apply also in this case and it is noted that if $|\lambda| > \alpha_0$ immediately after the initial time or if $|\lambda| > \alpha_f$ immediately before the final time, coasting in the planetary orbits will reduce J. For most purposes, however, it is not desired to allow terminal coasts since these really represent changing the launch and arrival dates from those desired.

Mass-Optimized Trajectories

Consider a spacecraft that is launched on an interplanetary trajectory. The launch vehicle places the spacecraft on the desired geocentric asymptote departing Earth and then separates from the spacecraft. Midcourse and planet-orbit injection maneuvers are accomplished using a spacecraft propulsion system. Because of staging between the launch vehicle and the spacecraft, plus the difference in specific impulse of the different propulsion systems, minimizing the total mission Δv does not maximize payload. Looking at the spacecraft alone, the final spacecraft mass in planet-orbit is

$$m_f = m_0 \exp\,(-\Delta v_T / v_j) \tag{17}$$

where m_0 is the injected spacecraft mass at the start of the transfer,

$$\Delta v_T = \sum_{k=2}^{n} |\Delta v_k| = \text{total spacecraft characteristic velocity}$$

and v_j is the spacecraft rocket jet velocity (specific impulse times Earth sea level gravitational acceleration). If m_0 were fixed, simply minimizing Δv_T would maximize m_f. But m_0 is generally a function of both the launch declination and the hyperbolic excess velocity, $m_0 = m_0(\delta, v_h)$. Now define a constant m_0^* where, e.g.,

$$m_0^* = m_0(\delta = 0, v_h = 0) \tag{18}$$

and write

$$m_f = m_0^* \exp(-(\Delta v_T + c)/v_j) \tag{19}$$

where c is a variable quantity to be determined. Expanding

$$m_f = m_0^* \exp(-c/v_j) \exp(-\ \Delta v_T / v_j) \tag{20}$$

Obviously choosing c such that

$$m_0^* \exp(-c/v_j) = m_0 \tag{21}$$

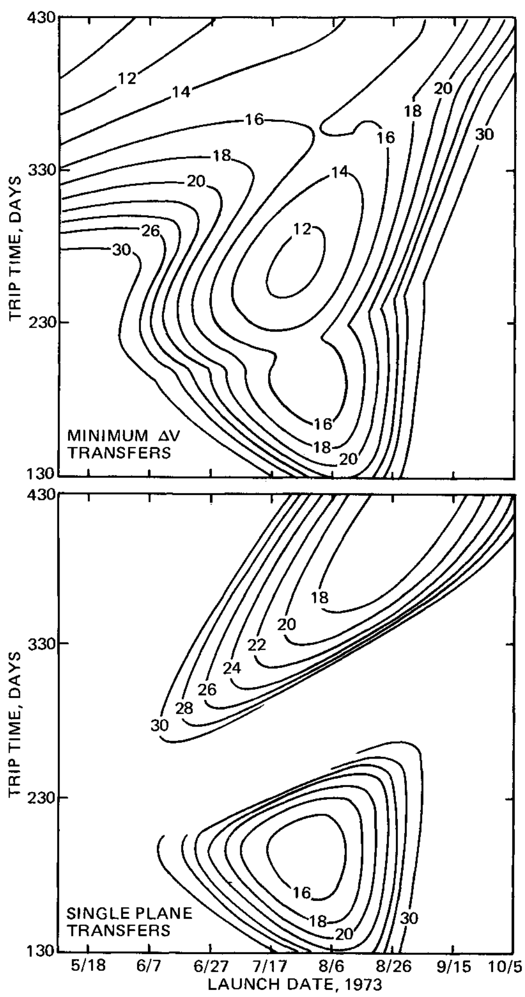

Fig. 4 C_3 of the geocentric asymptote, km²/sec².

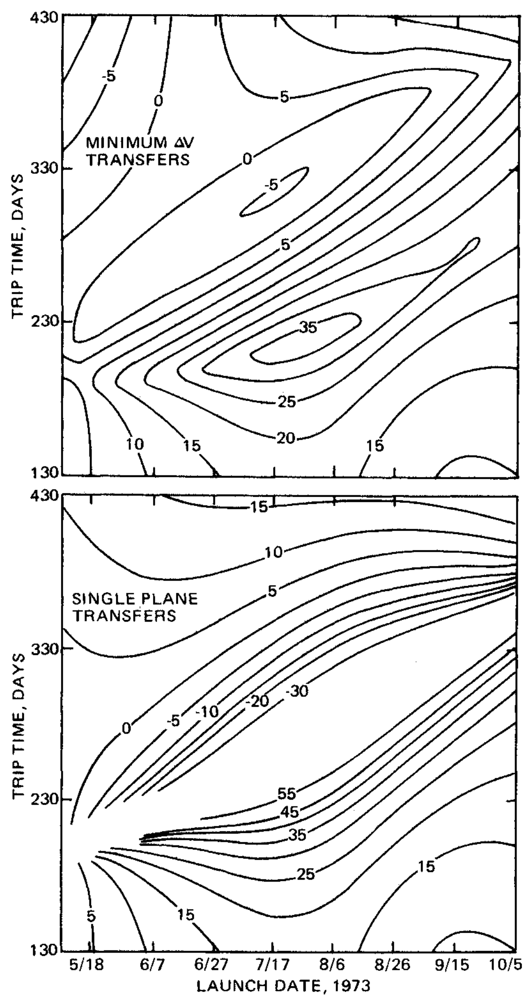

Fig. 5 Declination of the geocentric asymptote, deg.

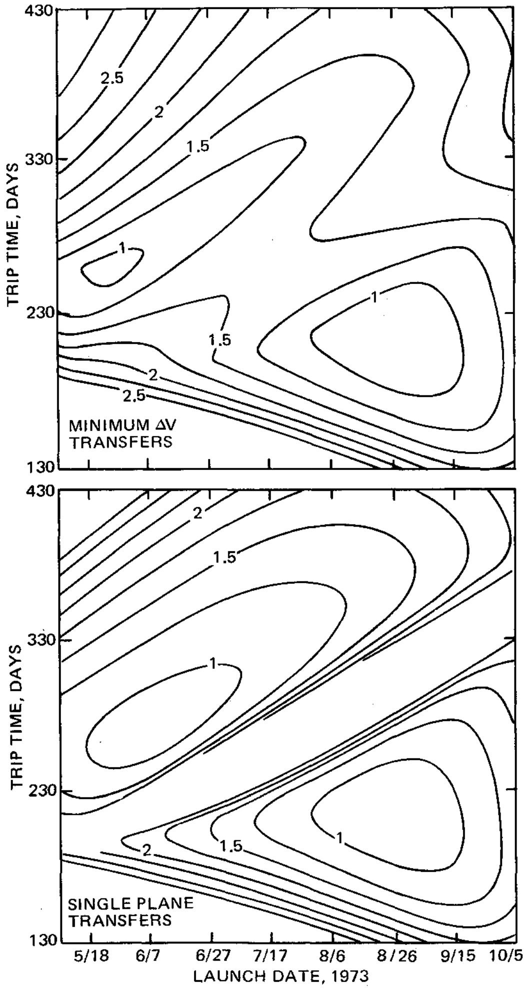

Fig. 6 Spacecraft Δv, km/sec.

makes Eq. (20) identical to Eq. (17). But m_0^* is a constant. Hence, to maximize m_f, it is necessary to minimize the sum $\Delta v_T + c$ where c is defined by Eq. (21)

$$c = -v_j \ln(m_0/m_0^*) \tag{22}$$

This problem is now identical to the minimum-impulse problem solved above. The sum $\Delta v_T + c$ is called the equivalent spacecraft characteristic velocity since it is the Δv which the spacecraft propulsion system would provide with an initial spacecraft mass m_0^* and a final spacecraft mass m_f.

Again, Eq. (9) yields the initial primer, λ_0. Now, however, since c depends on δ as well as v_h, λ_0 will undergo a rotation as well as a change in magnitude. Since $m_0 = m_0(\delta, v_h)$

$$\lambda_0 = \frac{\partial c}{\partial v_0^+} = -\frac{v_j}{m_0}\left[\frac{\partial m_0}{\partial v_h}\frac{\partial v_h}{\partial v_0^+} + \frac{\partial m_0}{\partial \delta}\frac{\partial \delta}{\partial v_0^+}\right] \tag{23}$$

In heliocentric-ecliptic, mean equinox of date coordinates, let v_0^+ be expanded into elements

$$v_0^+ = \begin{bmatrix} v_{0x} \\ v_{0y} \\ v_{0z} \end{bmatrix} \tag{24}$$

Then, δ is given by

$$\delta = \arctan x \tag{25}$$

where

$$x = \frac{v_{0x}\sin\epsilon + v_{0z}\cos\epsilon}{[v_{0x}^2 + (v_{0y}\cos\epsilon - v_{0z}\sin\epsilon)^2]^{1/2}} \tag{26}$$

and ϵ is the obliquity of the ecliptic. Using Eqs. (22, 23, 25, and 26), λ_0 becomes

$$\lambda_0 = -(v_j/m_0)[(\partial m_0/\partial v_0^+)\hat{\eta}_0 + (\partial m_0/\partial \delta)[1/(1+x^2)]\partial x/\partial v_0^+] \tag{27}$$

where $\hat{\eta}_0$ is a unit vector in the direction of the hyperbolic excess velocity, $\hat{\eta}_0 = \text{unit}\,(v_0^+ - v_0^-)$.

The derivatives $\partial x/\partial v_0^+$ are given in the Appendix and the derivatives $\partial m_0/\partial v_h$ and $\partial m_0/\partial \delta$ depend on the launch vehicle. Typically, the launch vehicle performance can be modeled to include constraints, daily launch windows, and any other desired effects. One might obtain, for example,

$$m_0 = a_1 + a_2\delta + a_3\delta^2 - a_4\ln(v_h^2 + a_5) \tag{28}$$

where the a_i are constants. Then, clearly

$$\partial m_0/\partial \delta = a_2 + 2a_3\delta, \quad \partial m_0/\partial v_h = 2a_4 v_h/(v_h^2 + a_5) \tag{29}$$

Now λ_0 can be computed and the trajectory optimization follows directly.

Notice in Eq. (27) that the term $(-v_j/m_0)(\partial m_0/\partial v_0^+)$ which multiplies $\hat{\eta}_0$, is a scalar factor equivalent to α. This term serves to scale λ_0 only. However, the second term in Eq. (27) causes a rotation in λ_0.

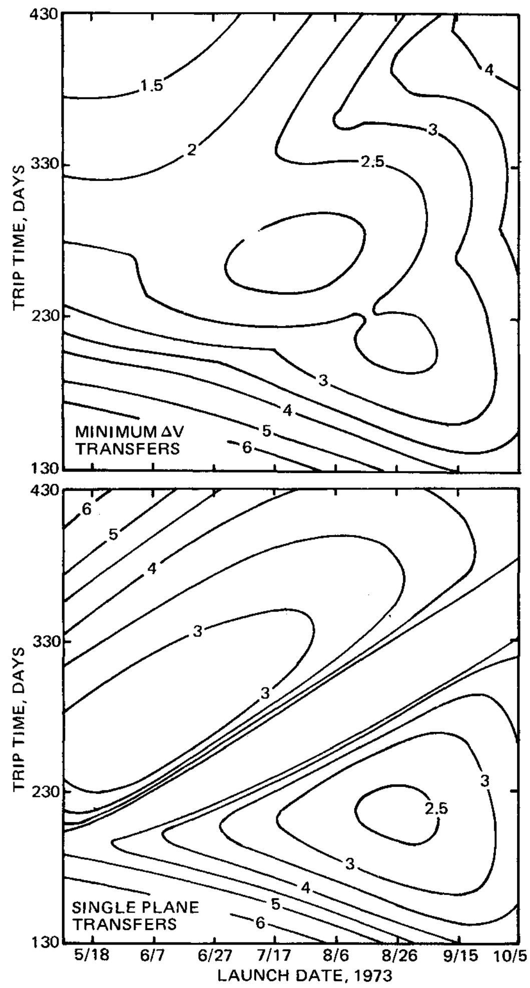

Fig. 7 Hyperbolic velocity at Mars, km/sec.

Results

Based on the theory developed in this paper, two computer programs have been written: Trajectory Optimization Program for Interplanetary Chemical Systems (TOPICS)[5] for computing minimum-Δv trajectories, and TOPICS-M,[13]

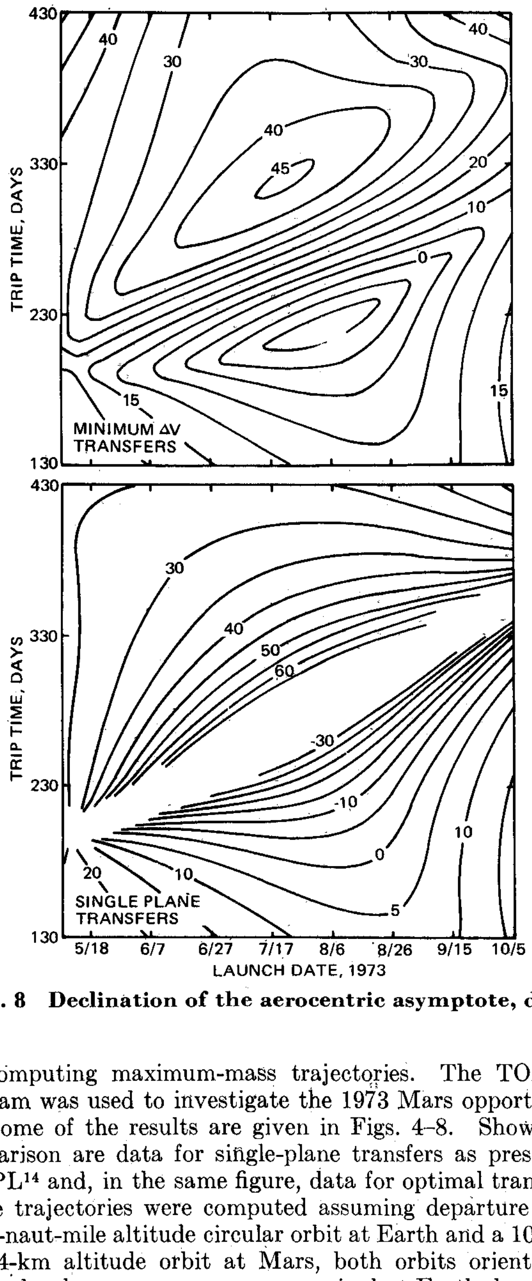

Fig. 8 Declination of the aerocentric asymptote, deg.

for computing maximum-mass trajectories. The TOPICS program was used to investigate the 1973 Mars opportunity and some of the results are given in Figs. 4–8. Shown for comparison are data for single-plane transfers as presented by JPL[14] and, in the same figure, data for optimal transfers. These trajectories were computed assuming departure from a 100-naut-mile altitude circular orbit at Earth and a $1000 \times 33{,}124$-km altitude orbit at Mars, both orbits oriented so that only planar maneuvers are required at Earth departure and Mars arrival. The launch vehicle advantages of using a broken-plane transfer are quite obvious, giving both reduced $C_3(v_h^2)$ and reduced declination. On the other hand, the spacecraft Δv, Δv_T, is increased slightly. Whether the broken-plane transfer offers advantages over the single-plane transfer for the total mission depends on the desired final spacecraft orbit and the specific impulse difference between the launch vehicle and the spacecraft.

Several mass-optimized trajectories, corresponding to the above minimum-impulse trajectories, were also computed using the TOPICS-M program. These generally showed a very small increase in final spacecraft mass over the minimum-impulse case—usually 1% or less.

To construct the curves shown, 961 optimal trajectories were computed on a CDC 6400 computer. The average CPU time was approximately three seconds per trajectory.

Conclusions

The method described affords a highly reliable and efficient scheme for computing minimum-impulse or maximum spacecraft mass trajectories. The scheme preserves the mathematical simplicity of the central-body, minimum-impulse problem and, in fact, with the inherent constraint preventing terminal coasts, actually improves convergence.

It is also shown that the use of optimal interplanetary trajectories can improve the Earth-launch geometry.

In general, the minimum-impulse transfer is nearly equal to the mass-optimal transfer and adequate results are usually obtained by determining the minimum-impulse transfer. However, for certain cases, payload improvements of 1–2% were obtained by the mass-optimal transfer compared to the minimum-impulse transfer—an amount that could be important for payload-limited launch vehicle/spacecraft configurations.

Appendix

This appendix gives the derivatives $\partial x/\partial v_0^+$. Let

$$\xi = [v_{0x}^2 + v_{0y}^2 \cos^2\epsilon + v_{0z}^2 \sin^2\epsilon - 2v_{0y}v_{0z} \cos\epsilon \sin\epsilon]^{1/2}$$

and

$$\beta = -(v_{0y} \sin\epsilon + v_{0z} \cos\epsilon)\xi^{-3}$$

Then

$$\partial x/\partial v_{0x} = v_{0x}\beta$$

$$\partial x/\partial v_{0y} = \beta(v_{0y} \cos^2\epsilon - v_{0z} \cos\epsilon \sin\epsilon) + (\sin\epsilon)/\xi$$

$$\partial x/\partial v_{0z} = \beta(v_{0z} \sin^2\epsilon - v_{0y} \cos\epsilon \sin\epsilon) + (\cos\epsilon)/\xi$$

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